Gravitation, rotation, and the profile of the Earth

(based on the book *Geophysical Theory* by Menke & Abbott)

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An isolated, non-rotating, self-gravitating fluid in empty space will take the form of a sphere. The Earth is fairly isolated, self-gravitating, and behaves as a fluid on geological time-scales. However, the Earth is rotating, as you probably noticed this morning.

It is also known from observations that the Earth is not a sphere, but is rather an *oblate spheroid*: its radius is larger at the equator and smaller at the poles. Why does the Earth take this shape, what is the amplitude of deviation from spherical, and how can we calculate it?

1 The profile by scaling argument

We know that the Earth is nearly a sphere, but we expect the rotation to cause a small equatorial bulge. The relevant forces are thus gravitation and centrifugal, with the latter causing a perturbation to the sphere and the former resisting it. These forces (per unit mass) scale as

$$F_{\text{gravity}} \sim \frac{GM}{R^2},$$
$$F_{\text{centrifugal}} \sim \omega^2 R,$$

where G is the gravitational constant, M is the mass of the Earth, R is the mean radius of the Earth, and ω is the rotation rate. We might expect the amplitude of deviation of the Earth from spherical to scale with the ratio of these forces:

$$\frac{R_{\text{equator}} - R_{\text{pole}}}{R} \sim \frac{\omega^2 R^3}{GM}.$$

Using $G = 7\times10^{-11}$ m^3 kg^{-1} s^{-2}, $M = 6\times10^{24}$ kg, $R = 6\times10^6$ m, and $\omega = 7\times10^{-5}$ radians s^{-1} we obtain a deviation amplitude of 0.25 per cent, or about 16 kilometres. In what follows, we compute this more rigorously.

2 Gravitation around a planet

Newton discovered that the gravitational force between two distant bodies is described by

$$f = \frac{GMm}{r^2}, \tag{1}$$

where M and m are the masses of the two bodies, r is the distance over which they are separated, and G is the gravitational constant. The force on a test particle with unit mass at a position $\boldsymbol{x}$ due to a point particle with mass M at position $\boldsymbol{x}_0$ is then given by the vector

$$\boldsymbol{f} = -\frac{GM}{|\boldsymbol{x} - \boldsymbol{x}_0|^3}(\boldsymbol{x} - \boldsymbol{x}_0). \tag{2}$$

The vector field $\boldsymbol{f}(\boldsymbol{x})$ is the gravitational field of the point mass.

2.1 Gravitational work and gravitational potential

A gravitational field can do work by moving a mass. Work is defined to be force times distance, or more precisely,

$$\Phi = -\int_C \boldsymbol{f} \cdot \hat{\boldsymbol{t}}\, \mathrm{d}s, \tag{3}$$

where Φ is the work done to move a unit mass along the curve C with unit tangent vector $\hat{\boldsymbol{t}}$. $\mathrm{d}s$ is an infinitesimal step along the curve.

For a point mass with a gravitational field given by (2), the equation for work takes a simpler form. This is because the gravitational field of the point mass depends only on distance and points in the direction along the line between the point-mass and the test-mass. Consider a curve C that is a closed loop in a gravitational field $\boldsymbol{f}$ resulting from a point mass. Using Stoke's theorem we have

$$\oint_C \boldsymbol{f} \cdot \hat{\boldsymbol{t}}\, \mathrm{d}s = \int_A \boldsymbol{\nabla} \times \boldsymbol{f}\, \mathrm{d}A, \tag{4}$$

where A is the area within the closed loop. We can take the curl of $\boldsymbol{f}$ in spherical coordinates as

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \det \begin{vmatrix} \frac{\hat{\boldsymbol{r}}}{r^2 \sin\theta} & \frac{\hat{\boldsymbol{\theta}}}{r \sin\theta} & \frac{\hat{\boldsymbol{\phi}}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ r\, f_r & 0 & 0 \end{vmatrix} = 0. \tag{5}$$

This result tells us that the gravitational field is *conservative*: motions that begin and end at the same point have no net work. Furthermore, it means the the work associated with bringing a test-mass in from infinity depends only on the final position of the test mass:

$$\Phi(r) = -\int_\infty^r \left(-\frac{GM\hat{\boldsymbol{r}}}{r^2}\right) \cdot \hat{\boldsymbol{r}}\, \mathrm{d}r = -\frac{GM}{r}. \tag{6}$$

Or, if the point-mass is at $\boldsymbol{x}_0$, the work done to bring a test mass to $\boldsymbol{x}$ is

$$\Phi(\boldsymbol{x}) = -\frac{GM}{|\boldsymbol{x} - \boldsymbol{x}_0|}. \tag{7}$$

By comparing equations (7) and (2) it should be clear that Φ is the *gravitational potential*: the potential energy per kilogram of test-mass. We can recover the gravitational force per unit mass by taking the gradient of the potential,

$$\boldsymbol{f} = -\boldsymbol{\nabla}\Phi. \tag{8}$$

The gravitational potential has the advantage that it is a scalar field, rather than a vector field.

2.2 Gravitational potential for continuous, heterogeneous bodies

We have the potential for a point mass, but what is the gravitational potential around a solid body with mass density $\rho(\boldsymbol{x})$? To derive an equation for this potential, we consider an arbitrary, closed surface in space S and a point mass M. We then consider the integral of the force normal to S:

$$\oint_S \boldsymbol{f} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S = \begin{cases} 0 & \text{for } M \text{ outside } S, \\ -4\pi GM & \text{for } M \text{ within } S, \end{cases} \tag{9}$$

where $\hat{\boldsymbol{n}}$ is an outward-pointing unit vector. This result can be shown by breaking the surface S into infinitesimal segments that are either normal to or parallel to $\hat{\boldsymbol{r}}$, which points radially away from the point mass. Applying Gauss' theorem gives

$$\int_V \nabla \cdot \boldsymbol{f}\, \mathrm{d}V = \begin{cases} 0 & \text{for } M \text{ outside } V, \\ -4\pi GM & \text{for } M \text{ within } V. \end{cases} \tag{10}$$

There is a special function that allows us to write the RHS of this equation without a 'case' statement: the *Dirac delta function*, $\delta(\boldsymbol{x} - \boldsymbol{x}_0)$. This function has the property that

$$\int_V \delta(\boldsymbol{x})\, \mathrm{d}V = \begin{cases} 0 & \text{for } \boldsymbol{x} \text{ outside } V, \\ 1 & \text{for } \boldsymbol{x} \text{ within } V. \end{cases} \tag{11}$$

Using equation (11) and recognising that the volume V is arbitrary (and can be taken to be arbitrarily small), we can rewrite (10) as

$$\nabla \cdot \boldsymbol{f} = -4\pi GM\delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{12}$$

for a point mass at $\boldsymbol{x}_0$. Using the definition of the gravitational potential, we can rewrite this as

$$\nabla^2\Phi = 4\pi GM\delta(\boldsymbol{x} - \boldsymbol{x}_0). \tag{13}$$

Now suppose that there are many point masses M_i at positions $\boldsymbol{x}_i$. Each occupies a volume $\mathrm{d}V_i$ and has a density $\rho(\boldsymbol{x}_i)\delta(\boldsymbol{x} - \boldsymbol{x}_i)$. By the linear superposition principle,

$$\begin{aligned} \nabla^2\Phi &= 4\pi G \sum_i \rho(\boldsymbol{x}_i)\delta(\boldsymbol{x} - \boldsymbol{x}_i)\mathrm{d}V_i \\ &= 4\pi G \int_V \rho(\boldsymbol{x}_0)\delta(\boldsymbol{x} - \boldsymbol{x}_0)\mathrm{d}V_0 \\ &= 4\pi G\rho(\boldsymbol{x}), \end{aligned} \tag{14}$$

where we have taken the limit of a continuous distribution of mass within the body. Equation (14) is Poisson's equation, and we must solve it to obtain the gravitational potential through the space within and around the body.

2.3 Aside: Green's function solution to Poisson's equation

We can obtain a simple and useful solution to Poisson's equation if we consider the case of an infinitesimal point mass M_i. Using what we learned previously about point-masses,

$$\mathrm{d}\Phi_i = -\frac{G\rho(\boldsymbol{x}_i)\, \mathrm{d}V_i}{r^2}. \tag{15}$$

We can again apply the superposition principle to sum the solutions for a family of point masses that make up a continuous body. Just as we did in equation (14), we integrate over point masses to obtain

$$\Phi(\boldsymbol{x}) = -G \int_V \frac{\rho(\boldsymbol{x}_0)}{|\boldsymbol{x} - \boldsymbol{x}_0|}\mathrm{d}\boldsymbol{x}_0, \tag{16}$$

where the integration is done in three dimensions, over the volume of the body,

$$\int_V \bullet\, \mathrm{d}\boldsymbol{x} \equiv \iiint \bullet\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3. \tag{17}$$

Equation (16) is actually a Green's function solution to the linear problem

$$\mathcal{L}[\Phi(\boldsymbol{x})] = \mathcal{F}(\boldsymbol{x}), \tag{18}$$

where $\mathcal{L} = \nabla^2$ is the linear operator and $\mathcal{F}(\boldsymbol{x}) = 4\pi G\rho(\boldsymbol{x})$ is the forcing function. The solution can then be written as

$$\Phi(\boldsymbol{x}) = \int_V \mathcal{F}(\boldsymbol{x}_0)\mathcal{G}(\boldsymbol{x}|\boldsymbol{x}_0)\, \mathrm{d}\boldsymbol{x}_0, \tag{19}$$

where $\mathcal{G}(\boldsymbol{x}|\boldsymbol{x}_0)$ is the Green's function solution at $\boldsymbol{x}$ for a delta-function source at $\boldsymbol{x}_0$.

2.4 Gravitational potential in and around a uniform sphere

The simplest approximation for an isolated planet is a sphere of uniform density ρ and radius R. By symmetry, the potential within and outside the planet must be spherically symmetric—it can depend only on the distance from the centre of the planet.

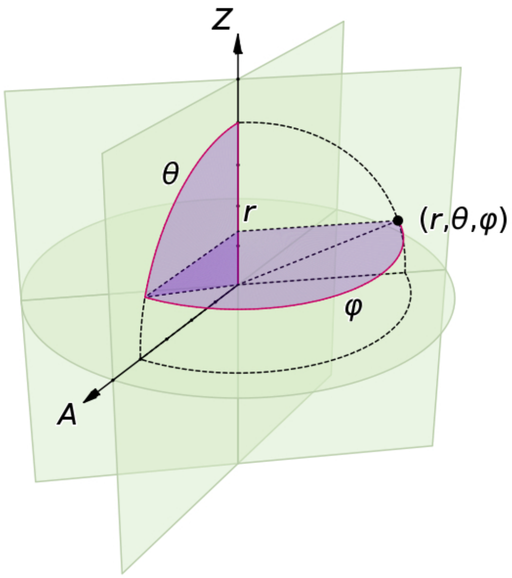

Figure 1: Spherical coordinate system (image from Wikipedia).

Recall that in spherical coordinates (Figure 1),

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right). \tag{20}$$

Hence Poisson's equation for the gravitational potential becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = \begin{cases} 0 & \text{outside the planet,} \\ 4\pi G\rho & \text{inside the planet.} \end{cases} \tag{21}$$

Which is a second-order ordinary differential equation (requiring two boundary conditions).

Inside the planet Within the planet, for $r < R$, we expect the potential to increase outward, as the amount of mass beneath increases. We therefore take the trial solution $\Phi(r) = cr^s$, where c and s are unknown constants. Inserting the trial solution into (21) we find that

$$cs(1+s)\frac{r^s}{r^2} = 4\pi G\rho. \tag{22}$$

To satisfy this equation, we need $s = 2$ and $c = 4\pi G\rho/6$. We must also remember to add to this all solutions to the homogeneous equation $\nabla^2\Phi = 0$, in unknown amounts:

$$\Phi_{\text{in}}(r) = \frac{4\pi G\rho}{6}r^2 + a_0 + \frac{b_0}{r}. \tag{23}$$

The term with b_0 must be zero because we don't expect a singularity at $r = 0$.

Outside the planet Outside of the planet, for $r > R$, the potential should decrease with radius and be spherically symmetrical. It can be shown that from outside, a uniform planet has the same potential as a point mass at its centre, with equal mass. Hence we have

$$\Phi_{\text{out}}(r) = c_0 + \frac{d_0}{r}. \tag{24}$$

Using the boundary condition $\Phi(r \to \infty) = 0$, we can take $c_0 = 0$.

Matching solutions at $r = R$ We require two more boundary conditions to solve for constants a_0 and d_0. These are found at the surface of the planet, where the potential and the gravitational field must be *continuous*. Hence we have

$$\Phi_{\rm in}(R) = \Phi_{\rm out}(R) \quad \text{and} \quad \left.\frac{\partial \Phi_{\rm in}}{\partial r}\right|_{r=R} = \left.\frac{\partial \Phi_{\rm out}}{\partial r}\right|_{r=R}. \tag{25}$$

The second of these boundary conditions implies that

$$d_0 = -\frac{4\pi R^3 \rho G}{3} = -GM, \tag{26}$$

while the first one gives us

$$a_0 = -2\pi\rho G R^2. \tag{27}$$

The full potential around an isolated sphere of uniform density is then

$$\Phi(r) = \begin{cases} -GM\dfrac{3R^2 - r^2}{2R^3} & \text{for } r \le R, \\ -\dfrac{GM}{r} & \text{for } r \ge R. \end{cases} \tag{28}$$

2.5 Gravitational potential around a non-uniform sphere

Planets are, in general, not spherically symmetrical, and so we need to find solutions that are applicable in the general case. Here we consider only the potential *outside* of a non-uniform planet, but this will suffice for present purposes. Outside of a planet, in open space, we have

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial \Phi}{\partial \phi}\right). \tag{29}$$

This linear equation can be solved by the method of separation of variables. We assume the solution has the following form

$$\Phi(r, \theta, \phi) = A(r)\, B(\theta)\, C(\phi). \tag{30}$$

Substituting this into Laplace's equation and following the usual procedure gives the three equations

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}A}{\mathrm{d}r}\right) - n(n+1), \tag{31a}$$

$$0 = (1-\zeta^2)\frac{\mathrm{d}^2 B}{\mathrm{d}\zeta^2} - 2\zeta\frac{\mathrm{d}B}{\mathrm{d}\zeta} + \left[n(n+1) - \frac{m^2}{1-\zeta^2}\right] B, \tag{31b}$$

$$0 = \frac{\mathrm{d}^2 C}{\mathrm{d}\phi^2} + m^2 C, \tag{31c}$$

where $\zeta = \cos\theta$ and m, n are constants that arise in the separation procedure. The third of these has the solution

$$C(\phi) \propto \mathrm{e}^{\pm i m \phi}, \tag{32}$$

And we see that m must be an integer such that this function is periodic with period 2π.

For the special case of axial symmetry (when $m = 0$), equation (31b) has solutions that are Legendre polynomials with $\cos\theta$ as their argument,

$$B(\theta) \propto P_n(\cos\theta). \tag{33}$$

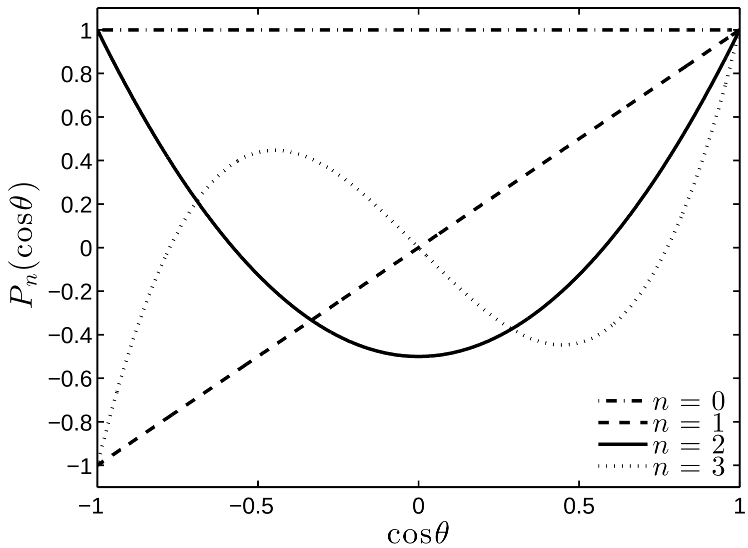

Figure 2: The first four Legendre polynomials.

The first four Legendre polynomials are shown in Figure 2.

Finally, equation (31a) gives rise to two solutions that are

$$A(r) \propto r^n \quad \text{and} \quad A(r) \propto r^{-(n+1)}. \tag{34}$$

Putting these together as in (30), for an axially symmetrical planet, gives

$$\Phi(r, \theta) = \sum_{n=0}^{\infty} \left[a_n r^n + \frac{b_n}{r^{n+1}} \right] P_n(\cos\theta), \tag{35}$$

where a_n and b_n are coefficients to be determined. For completeness, the solution without axial symmetry is

$$\Phi(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[a_{mn} r^n + \frac{b_{mn}}{r^{n+1}} \right] P_n^m(\cos\theta) \mathrm{e}^{im\phi}. \tag{36}$$

3 Rotation and centrifugal force

The rotation of the Earth gives rise to an acceleration of material radially inward. In the non-inertial rotating reference frame, this is experienced as a centrifugal force, radially outward. The magnitude of this force depends the angle from the pole of rotation and a vector from the centre of the Earth to the material point. The centrifugal force is greatest at the equator and least at the pole. Combined with the gravitational force, this gives rise to the equatorial bulge of the Earth's profile.

The gravitational field and the centrifugal force combine to create an *effective gravitational field*. This field is associated with a potential that *not* spherically symmetrical. An isosurface of this potential is called the *geoid*. Over a very long time, the solid Earth deforms such that its surface closely matches an equipotential surface. Deviations from this match represent lateral gradients in potential energy, which can drive flow.

The velocity of a vector $\boldsymbol{x}$ rotating around a fixed axis at a rate $\boldsymbol{\omega}$ is given by $\boldsymbol{\omega} \times \boldsymbol{x}$; the acceleration of that vector is given by $\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{x}$. Since the centrifugal force (per unit mass) is a fictitious force that opposes this acceleration, it is given by

$$\boldsymbol{f}_c = -\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{x}. \tag{37}$$

In Cartesian coordinates, with the rotation vector pointed in the z-direction, we have $\boldsymbol{\omega} = \omega\hat{\boldsymbol{k}}$. We can choose a value of $\boldsymbol{x}$ that is in the x-z plane, such that $\boldsymbol{x} = [r\sin\theta, 0, r\cos\theta]^T$, where θ is the angle between $\hat{\boldsymbol{k}}$ and $\boldsymbol{x}$. In this coordinate system, the centrifugal force per unit mass becomes

$$\begin{aligned} \boldsymbol{f}_c &= \omega^2 r \sin\theta\,\hat{\boldsymbol{\imath}} \\ &= \omega^2 x\,\hat{\boldsymbol{\imath}}, \end{aligned} \tag{38}$$

where $\hat{\boldsymbol{\imath}}$ is a unit-vector in the x-direction.

The centrifugal potential is the function Φ_c that satisfies

$$\boldsymbol{f}_c = -\boldsymbol{\nabla}\Phi_c. \tag{39}$$

By inspection of (38), we can see that

$$\begin{aligned} \Phi_c &= -\frac{1}{2}\omega^2 x^2 \\ &= -\frac{1}{2}\omega^2 r^2 \sin^2\theta. \end{aligned} \tag{40}$$

In what follows, we will use Legendre polynomials as a basis for solutions to Poisson's equation for the potential. It is therefore useful to express (40) in terms of Legendre polynomials. This can be accomplished by using the identity

$$\sin^2\theta = \frac{2}{3}\left[1 - P_2(\cos\theta)\right]. \tag{41}$$

Substitution into (40) gives

$$\Phi_c = -\frac{1}{3}\omega^2 r^2 + \frac{1}{3}\omega^2 r^2 P_2(\cos\theta). \tag{42}$$

Figure 2 shows the first four Legendre polynomials.

4 Equilibrium shape of the rotating Earth

The potential of the effective gravitational field Φ^{eff} is the sum of the gravitational potential and the centrifugal potential. The geoid is the isosurface corresponding to the radius of the Earth r_E. We will now assume that the radius of the Earth varies with colatitude θ. The solution that we seek, $r_E(\theta)$, is given implicitly by

$$\Phi^{\text{eff}} = \Phi[r = r_E(\theta), \theta] + \Phi_c[r = r_E(\theta), \theta] = \text{const}. \tag{43}$$

To solve this problem for r_E we must make some simplifications. The first is to assume that the departure from sphericity is small. Observationally, this is a good approximation for the Earth. We will also assume that the Earth is constant density, which is not such a good assumption, but it simplifies the problem considerably.

We can then represent the radius of the Earth in terms of Legendre polynomials as

$$r_E(\theta) = R\left[1 + \sum_{n=1}^{\infty} c_n P_n(\cos\theta)\right]. \tag{44}$$

Here R is the mean radius and $c_n \ll 1$ are undetermined coefficients. The Legendre polynomials are symmetric around the rotation axis, which is well-suited to our problem.

We can approximate the gravitational potential both within and outside the Earth with Legendre polynomials. Again these play the role of a small perturbation to the spherically symmetrical solution. Hence we have

$$\Phi^{\text{eff}}_{\text{in}}(r,\theta) = -GM\frac{3R^2 - r^2}{2R^3} + \sum_{n=1}^{\infty} b_n \left(\frac{r}{R}\right)^n P_n(\cos\theta), \tag{45}$$

$$\Phi^{\text{eff}}_{\text{out}}(r,\theta) = -\frac{GM}{r} + \sum_{n=1}^{\infty} a_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta), \tag{46}$$

where a_n and b_n are sets of coefficients, to be determined. The factors $(r/R)^n$ and $(R/r)^{n+1}$ come from the two general solutions to radial component of Laplace's equation, equation (31a).

To solve the problem we must determine a_n, b_n, and c_n. This requires the use of three conditions, based on the physics of the problem. These are

1. Gravitational potential is continuous across the surface of the Earth.
2. Normal derivative of the gravitational potential is continuous across the surface of the Earth.
3. The surface of the Earth is an equipotential (the geoid).

The first two conditions are problematic because the surface of the Earth is no longer spherical and normal to the radial direction. To deal with this we make another approximation: we assume that the boundary is spherical, but containing an infinitesimal shell with variable density, representing the deviation from sphericity. Since $(R - r_E)/R \ll 1$, we can quantify the mass of the bulge as a spherical surface at $r = R$ with a density ρ' that depends on colatitude θ. This is written

$$\rho'(r,\theta) = \rho\,[r_E(\theta) - R]\,\delta(r - R), \tag{47}$$

where the Dirac delta function limits the density to exist in a spherical shell with radius R. Note that ρ' is actually negative near the poles, where $r_E < R$.

Under this approximation, we can easily apply the first condition, that the gravitational potential is continuous from within to outside of our model of the Earth. Doing so gives

$$\begin{gathered} \Phi^{\text{eff}}_{\text{in}}(r = R,\theta) = \Phi^{\text{eff}}_{\text{out}}(r = R,\theta), \\ -\frac{GM}{R} + \sum_{n=1}^{\infty} b_n P_n(\cos\theta) = -\frac{GM}{R} + \sum_{n=1}^{\infty} a_n P_n(\cos\theta), \end{gathered} \tag{48}$$

which shows that $a_n = b_n$.

Because of the assumed discontinuity in the density at the surface of the Earth, the radial gradient of the gravitational potential is no longer continuous there. The jump in gradient is due to the delta-function component of the forcing (density) function, and hence we have

$$\left.\frac{\partial\Phi^{\text{eff}}_{\text{out}}}{\partial r}\right|_{(r=R,\theta)} - \left.\frac{\partial\Phi^{\text{eff}}_{\text{in}}}{\partial r}\right|_{(r=R,\theta)} = 4\pi G\rho\,[r_E(\theta) - R]. \tag{49}$$

Substitution of equations (44), (45), and (46) with $b_n = a_n$ gives

$$\left[\frac{GM}{R^2} - \sum_{n=1}^{\infty}(n+1)\frac{a_n}{R}P_n(\cos\theta)\right] - \left[\frac{GM}{R^2} + \sum_{n=1}^{\infty} n\frac{a_n}{R}P_n(\cos\theta)\right] = 4\pi G\rho R\sum_{n=1}^{\infty} c_n P_n(\cos\theta). \tag{50}$$

This equation tells us that

$$a_n = -\frac{4\pi G\rho R^2}{2n+1} c_n, \\ = -\frac{3GM}{(2n+1)R} c_n. \tag{51}$$

So this leaves us with just one set of unknown coefficients, and one final condition to be used to solve for them. Taking the r_E to be a surface of constant potential means that

$$\text{const} = \Phi(r_E, \theta) + \Phi_c(r_E, \theta), \\ = -\left[\frac{GM}{r_E} + \sum_{n=1}^{\infty} \frac{3GM}{(2n+1)R} c_n \left(\frac{R}{r_E}\right)^{n+1} P_n(\cos\theta)\right] + \left[\frac{\omega^2 r_E^2}{3}[P_2(\cos\theta) - 1]\right], \tag{52}$$

where we have used equations (46) and (42) for the gravitational and centrifugal potentials, respectively.

The mathematical problem, now, is to substitute $r_E(\theta) = R[1 + \sum c_n P_n(\cos\theta)]$ into (52), and solve for c_n. First we recall that $c_n \sim \mathcal{O}(\epsilon) \ll 1$ and use the binomial series expansions

$$(1+\epsilon)^{-1} \approx 1 - \epsilon + \mathcal{O}(\epsilon^2), \tag{53a}$$

$$(1+\epsilon)^{2} \approx 1 + 2\epsilon + \mathcal{O}(\epsilon^2). \tag{53b}$$

We retain only the leading-order terms in the centrifugal force (i.e. we neglect terms in $\omega^2 c_n$). The perturbation to the centrifugal force from the equatorial bulge is small enough to be quantitatively irrelevant. With these simplifications we have

$$-\frac{GM}{R}\left[1 - \sum_{n=1}^{\infty} c_n P_n(\cos\theta)\right] - \sum_{n=1}^{\infty} \frac{3GM}{(2n+1)R} c_n P_n(\cos\theta) + \left[\frac{\omega^2 R^2}{3}[P_2(\cos\theta) - 1]\right] = \text{const}. \tag{54}$$

Factoring out the Legendre polynomial we have

$$\sum_{n=1}^{\infty} \left\{\left[\frac{GM}{R} - \frac{3GM}{(2n+1)R}\right] c_n + \frac{1}{3}\delta_{2n}\omega^2 R^2\right\} P_n(\cos\theta) = \text{const}. \tag{55}$$

For the left-hand side of this equation to equal a constant, independent of θ, the quantity in the brackets must equal zero for all n. Thus we have $c_n = 0$ except for c_2, which solves

$$\left[\frac{GM}{R} - \frac{3GM}{(2n+1)R}\right] c_2 = -\frac{1}{3}\omega^2 R^2, \tag{56}$$

and so

$$c_2 = -\frac{5\omega^2 R^3}{6GM}. \tag{57}$$

Finally, substituting this result into equation (44) we obtain the solution

$$r_E \approx R\left[1 - \frac{5\omega^2 R^3}{6GM} P_2(\cos\theta)\right]. \tag{58}$$

Some remarks about this:

- The amplitude of the perturbation (given by c_2) is approximately the ratio the centrifugal force at the surface of the Earth, $w^2 R$, to the gravitational force there, GM/R^2.

- Inserting numbers into (58) we find that at the equation, we expect a bulge of 8 km at the equator, while the radius at the pole is reduced by 16 km. The total predicted difference is 24 km, which comparable to the observed difference of 20 km.

- An ellipsoid is given by the formula

$$\begin{aligned} r(\theta) &= R\left[\frac{(1-e)^2}{(1-e)^2\sin^2\theta+\cos^2\theta}\right]^{1/2}, \\ &= R\left[\left(1+\frac{e}{3}\right)+\frac{2}{3}eP_2(\cos\theta)\right]+\mathcal{O}(e^2), \end{aligned} \tag{59}$$

  where e is the *ellipticity*. When $e=0$, $r=R$ and we recover the shape of a sphere. The second line is an Taylor series expansion for $e \ll 1$, truncated at first order. Comparison of equation (59) with (58) shows that the shape of the Earth is well-approximated by an ellipsoid.