Flow over sinusoidal, porous sandbed

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Contents

1 Introduction

Due to water flow over the oceanic sediment the porous seabed is often rippled. These obstructions interact with the flow causing pressure gradients to build up, which drive flow through the sediment itself. Similar to surface wave induced flow within the sediment [Shum, 1992, 1993], this is yet another mechanism which facilitates exchange of minerals or Oxygen or there like with the water above[Rutherford and Boyle, 1995, Huettel et al., 1996, Evrard et al., 2012]. The flow considered is very similar to wind interacting with dunes, which also drives a flow through the porous sand and for example transports humidity into the sand [Louge et al., 2010]

So far this flow has been studied experimentally Savant et al. [1987], Thibodeaux and Boyle [1987], Louge et al. [2010], Musa et al. [2013] and numerically Meysman et al. [2007] there have been no attempts at finding an analytical solution for this flow.

There are several decisions to be made for setting up a model. One is whether to include turbulence or not and if Reynolds numbers are assumed low or high. The situation on the continental shelf differs in this respect from the situation of oceanic currents. Kuzan et al. [1989] found that for a solid wavy interface with the flow parameters as we would expect them on the continental shelf, there is most likely flow separation and hence turbulence. Another is the form of the boundary condition at the permeable interface. The boundary may be realised by the following approaches:

- Slip condition as described by Beavers and Joseph [1967] with Darcy's law within the porous medium

- A Brinkman layer within the porous medium by using including the Brinkman term to the Darcy equation, together with continuity of the flow variables

- A solid interface with Darcy flow in the porous medium driven by the pressure at the boundary.

- A solid boundary as above, but including roughness (only applicable for turbulent flows)

Here we assumed a flow at infinite Reynolds numbers, i.e. Euler flow or potential flow. In this case we can only impose a kinetic boundary condition which assumes a rigid wall. The plan is to perturb the found solution, assuming high but finite Reynolds numbers and small slopes, and impose the Beavers-Joseph boundary condition on this perturbed flow. This would be a solution of potential flow with a friction boundary layer. This still might severely limit the applicability of the model as turbulence was ignored. The set-up is depicted in Figure 1

The plan of work progress is:

- solve inviscid irrotational flow in the fluid assuming a rigid boundary

- obtain the streamline-coordinates (potential function and stream function)

- add a boundary layer, i.e. perturb and assume:

  - standard boundary layer character (gradient mainly perpendicular)
  - only a small perturbation to the inviscid solution

- obtain a formula for the perturbed streamfunction and boundary pressure

- match series coefficients by using full Beavers-Joseph condition

- use perturbed pressure at boundary to obtain corrected Darcy Flow

As an alternative approximation the situation is analysed employing the slow viscous assumption of Stokes flow.

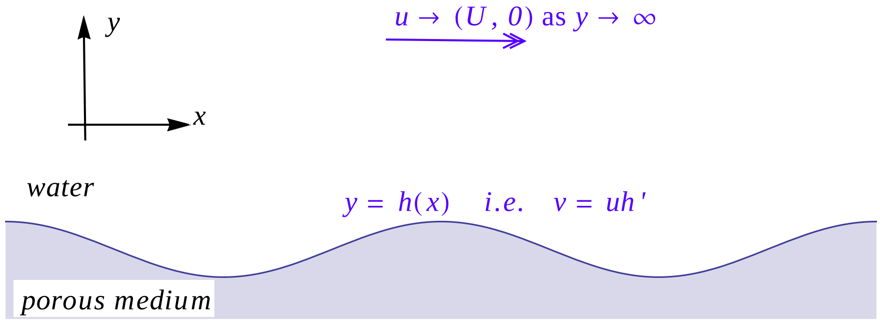

Figure 1: Problem analysed

2 Potential flow past a rigid, sinusoidal bed

As a first approximation, the flow is assumed to be inviscid. Having no initial vorticity we get potential flow, i.e. we can write $\boldsymbol{u} = \boldsymbol{\nabla}\phi$ for some function $\phi(x, y)$. Incompressibility of the flow gives Laplace's equation (1a). The boundary conditions are the far field condition of exterior flow $\boldsymbol{u} = (U, 0)$ (1b) and to leading order a stationary boundary at the sinusoidal surface $y = h(x) = \epsilon/k\cos(kx)$ (1c).

$$\begin{aligned}
&\text{Velocity Potential:} && \boldsymbol{u} = \boldsymbol{\nabla}\phi, && && \text{(1a)}\\
&\text{Continuity Equation:} && \nabla^2\phi = 0, && && \text{(1b)}\\
&\text{Far Field Condition:} && \phi \to Ux && \text{as} \quad y \to \infty, && \text{(1c)}\\
&\text{Stationary Boundary:} && \phi_y = h'(x)\phi_x && \text{on} \quad y = h(x) && \text{(1d)}
\end{aligned}$$

First we non-dimensionalise the equations by writing $x^* = kx$, $y^* = ky$, $\phi^* = kU^{-1}\phi$ and $h^* = k\epsilon^{-1}h = \cos(x^*)$. This gives a new, easier system of equations (2) (the asterisks denote dimensionless quantities)

$$\nabla^{*2}\phi^* = 0, \tag{2a}$$

$$\phi^* \to x^* \quad \text{as} \quad y^* \to \infty, \tag{2b}$$

$$\phi^*_{y*} = \epsilon h^{*\prime}\phi^*_{x*} \quad \text{on} \quad y^* = \epsilon h^*, \tag{2c}$$

From now on the asterisks are left out for better readability. An asymptotic solution can be obtained by expanding ϕ (3a) in the slope ϵ and by moving the boundary condition (3b) to $y = 0$ using a Taylor series. These then give an iterative procedure to calculate a series for the potential (4) and (5).

$$\phi = \phi^{(0)} + \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \dots, \tag{3a}$$

$$\phi_y + \epsilon h\phi_{yy} + \frac{1}{2}\epsilon^2 h^2\phi_{yyy} + \dots = \epsilon h'\phi_x + \epsilon^2 h' h\phi_{xy} + \dots \quad \text{on} \quad y = 0 \tag{3b}$$

Substituting (3a) into (3b) allows to look at the boundary condition to varying orders in ϵ.

$$\begin{aligned}
&\mathcal{O}(1): && \phi_y^{(0)} = 0, && \text{(4a)}\\
&\mathcal{O}(\epsilon): && \phi_y^{(1)} = -h\phi_{yy}^{(0)} + h'\phi_x^{(0)}, && \text{(4b)}\\
&\mathcal{O}(\epsilon^2): && \phi_y^{(2)} = -h\phi_{yy}^{(1)} - \frac{1}{2}h^2\phi_{yyy}^{(0)} + h'\phi_x^{(1)} + h'h\phi_{xy}^{(0)}, && \text{(4c)}\\
& && \vdots && \\
&\mathcal{O}(\epsilon^n): && \phi_y^{(n)} = \sum_{i=0}^{n-1}\frac{h^i}{i!}\left(h'\frac{\partial^{i+1}}{\partial y^i\partial x} - \frac{h}{i+1}\frac{\partial^{i+2}}{\partial y^{i+2}}\right)\phi^{(n-1-i)} && \text{(4d)}
\end{aligned}$$

The solution for the velocity potential can be easily seen as the imaginary part of a complex, analytical function $f(z)$ (6a). From the analysis of complex analytical functions we know that both the real and imaginary part satisfy Laplace's equation. Via the Cauchy-Riemann equations we see that the real part gives the non-dimensional streamfunction $\psi(x, y)$ (6c).

The first few explicit calculations (5) of the series in ϕ illustrate the procedure to obtain all terms.

$$\begin{aligned}
& & & \phi^{(0)} = x, && \text{(5a)}\\
&\phi_y^{(1)}|_{y=0} = -\sin(x) && \Rightarrow && \phi^{(1)} = \sin(x)\mathrm{e}^{-y} && \text{(5b)}\\
&\phi_y^{(2)}|_{y=0} = -\cos(x)\sin(x) - \sin\cos(x) = -\sin(2x), && \Rightarrow && \phi^{(2)} = \frac{1}{2}\sin(2x)\mathrm{e}^{-2y} && \text{(5c)}\\
&\vdots & & & \vdots &&
\end{aligned}$$

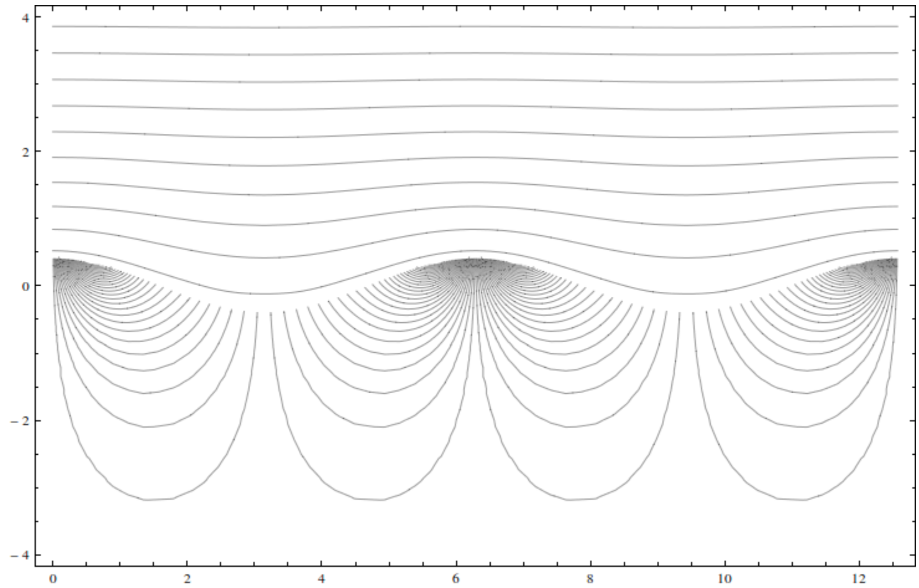

Figure 2: Streamlines of the solution in the water and in the porous sediment

$$
\begin{aligned}
f(z) &= \mathrm{i}z + \epsilon \mathrm{e}^{\mathrm{i}z} + \frac{\epsilon^2}{2}\mathrm{e}^{2\mathrm{i}z} + \frac{\epsilon^3}{8}\left(3\mathrm{e}^{3\mathrm{i}z} + \mathrm{e}^{\mathrm{i}z}\right) + \dots, && \text{(6a)}\\
\phi(x, y) &= \operatorname{Im} f(x + \mathrm{i}y) = x + \epsilon \sin(x)\mathrm{e}^{y} + \dots, && \text{(6b)}\\
\psi(x, y) &= -\operatorname{Re} f(x + \mathrm{i}y) = y - \epsilon \cos(x)\mathrm{e}^{y} + \dots && \text{(6c)}
\end{aligned}
$$

Now Bernoulli's equation is applied to obtain the pressure at the boundary. Note that the pressure is non-dimensionalised by the group ρU^2. As expected this scaling shows that the hydrostatic pressure is a effect of order $\mathcal{O}(\epsilon)$. We get a pressure in phase with the ripples, which is not supported by experiments, but is due to the simplifcation of ignoring the friction boundary layer and turbulence.

$$
\begin{aligned}
\text{const} &= p + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{gy}{U^2 k} && \text{(7a)}\\
p_{bdy}(x) &= \text{const} - \frac{g\epsilon}{U^2 k}\cos(x) - \frac{1}{2}\left(1 + 2\epsilon\cos(x) + \dots\right) && \text{(7b)}\\
&= -\left(1 + \frac{g}{U^2 k}\right)\epsilon\cos(x) + \dots, \quad \text{wlog const} = \frac{1}{2}
\end{aligned}
$$

3 Pressure induced Darcy flow

To find the flow in the sediment we use Darcy's Law in a non-dimensional form, with the pressure from the fluid solution and decay at $y = -\infty$:

$$
\begin{aligned}
\boldsymbol{u}_D &= \mathcal{D}a\mathcal{R}e\boldsymbol{\nabla}p && \text{(8a)}\\
\mathcal{D}a &= \kappa k^2 && \text{(8b)}\\
\mathcal{R}e &= \frac{U}{k\nu} && \text{(8c)}
\end{aligned}
$$

We additionally have continuity $\boldsymbol{\nabla}\cdot\boldsymbol{u}_D = 0$ which gives Laplace's equation $\nabla^2 p = 0$.

Again expand as a series in ϵ

$$p = p^{(0)} + \epsilon \cdot p^{(1)} + \epsilon^2 \cdot p^{(2)} + \dots \tag{9}$$

and match orders of ϵ

$$p^{(0)} + \epsilon \cdot p^{(1)} + \epsilon^2 \cdot p^{(2)} + \dots = p_{bdy}(x) = -\underbrace{\left(1 + \frac{g}{U^2 k}\right)}_{=\beta} \epsilon \cos(x) + \dots \tag{10}$$

Again this can be solved by means of a complex function, giving a potential and streamfunction for the flow velocity inside the sediment.

$$f(z) = \underbrace{\mathcal{D}a\mathcal{R}e\beta\epsilon}_{=A} \mathrm{e}^{-\mathrm{i}z} + \dots \tag{11a}$$

$$\phi(x, y) = \operatorname{Re} f(x + \mathrm{i}y) = A\cos(x)\mathrm{e}^y + \dots = \mathcal{D}a\mathcal{R}e \cdot p(x, y) \tag{11b}$$

$$\psi(x, y) = \operatorname{Im} f(x + \mathrm{i}y) = -A\sin(x)\mathrm{e}^y + \dots \tag{11c}$$

4 Physically interesting quantities

To obtain physical interesting quantities like flow times and fluxes we restrict ourselves to first order in ϵ. Also we assume a sinusoidal solution for the streamfunction with amplitude $A = \mathcal{O}(\epsilon)$ and possible phase shift θ.

$$\text{streamfunction}: \quad \psi = -A\sin(x + \theta)\mathrm{e}^y, \tag{12a}$$

$$\text{flow velocity}: \quad \boldsymbol{u}_D = (-\sin(x + \theta), \cos(x + \theta)) \cdot A\mathrm{e}^y, \tag{12b}$$

$$\text{outwards surface normal}: \quad \boldsymbol{n} = (\epsilon\sin(x), 1) \tag{12c}$$

We will require two scalar path integrals. The first one being an infinitesimal part of the sand-water boundary. The line element ds is approximated to first order in ϵ.

$$\Gamma_{x^*}: \ \boldsymbol{x}(\lambda) = (\lambda, \epsilon\cos(\lambda)) \quad \lambda \in [x^*, x^* + \delta x^*], \tag{12da}$$

$$\int_{\Gamma_{x^*}} \cdot \ \mathrm{d}\, s = \int_{x^*}^{x^* + \delta x^*} \cdot \sqrt{1 + \epsilon^2 \sin^2(\lambda)} \ \mathrm{d}\,\lambda \approx \int_{x^*}^{x^* + \delta x^*} \cdot \ \mathrm{d}\,\lambda, \tag{12db}$$

The second one being the integral along the individual pathlines, which are equivalent to streamlines and streaklines for a stationary flow like in this case. Hence the path equation is derived from $\psi_0(x^*) = \text{const} = -A\sin(x + \theta)\mathrm{e}^y$. The end points x^* and x^{**} come from the intersection of this with $y = \epsilon\cos(x)$ which to leading order gives the equation $\psi_0(x) \approx -A\sin(x + \theta)$.

$$\gamma_{x^*}: \ \boldsymbol{x}(\lambda) = \left(\lambda, \ln\left(\frac{\psi_0(x^*)}{A}\csc(\lambda + \theta)\right)\right), \tag{12ea}$$

$$\int_{\gamma_{x^*}} \cdot \ \mathrm{d}\, s = \int_{x^*}^{x^{**}} \cdot \ \csc^2(\lambda + \theta) \ \mathrm{d}\,\lambda, \tag{12eb}$$

$$x^{**} = \pi - x^* - 2\theta + 2N\pi \quad N \in \mathbb{Z} \tag{12ec}$$

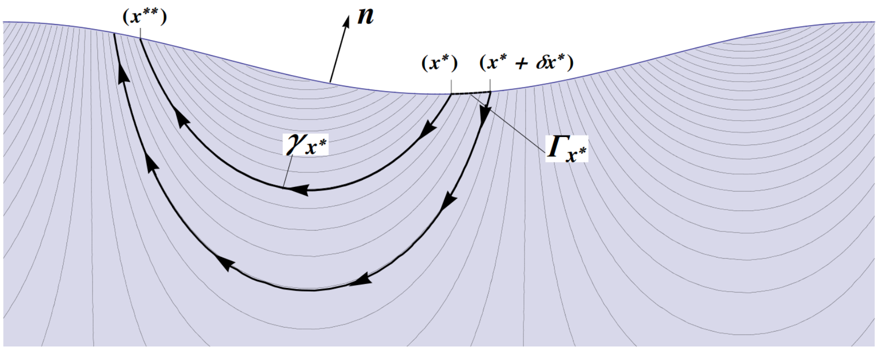

Figure 3: Streamlines of the solution in the water and in the porous sediment

Using this we can get some physically interesting quantities, all per unit length in z-direction and per wave length. First, the flux Φ through an infinitesimal part of the boundary

$$\begin{aligned}\Phi\left(x^*\right) &= \int_{\Gamma_{x^*}} \boldsymbol{u}_D \cdot (-\boldsymbol{n}) \ \mathrm{d}\, s = \int_{x^*}^{x^*+\delta x^*} \left(-A\cos(\lambda+\theta) + \mathcal{O}\left(A\epsilon\right)\right) \ \mathrm{d}\,\lambda \\ &= -A\cos(x^*)\delta x^* + \mathcal{O}\left(A\epsilon + A\left(\delta x^*\right)^2\right)\end{aligned} \tag{6}$$

Second, the flow time T for a specific entry point x^*. Using the absolute value we do not need to fix the value of x^{**} as the periodicity with $2N\pi$ only affects the sign, i.e. set $N = 0$.

$$\begin{aligned}T\left(x^*\right) &= \int_{\gamma_{x^*}} \left|\boldsymbol{u}_D\right|^{-1} \ \mathrm{d}\, s \approx \left|\int_{x^*}^{x^{**}} \frac{1}{A}\mathrm{e}^{-y}\csc^2(\lambda+\theta) \ \mathrm{d}\,\lambda\right| \\ &\approx \left|\int_{x^*}^{x^{**}} \frac{\sin(\lambda+\theta)}{\psi_0\left(x^*\right)}\csc^2(\lambda+\theta) \ \mathrm{d}\,\lambda\right| = \left|\frac{1}{\psi_0\left(x^*\right)}\left[\ln\left(\tan\left(\frac{\lambda+\theta}{2}\right)\right)\right]_{x^*}^{x^{**}}\right| \\ &\approx \frac{1}{A}\left|\csc\left(x^*+\theta\right)\right| \cdot \ln\left(\tan^2\left(\frac{x^*+\theta}{2}\right)\right)\end{aligned} \tag{7}$$

Finally these two combined give the rate of added particles to the main water body due to washing out within the sand. Where we are given some $\Delta c\left(T, c_0\right)$ describing the increase in concentration of water with a effective concentration c_0 over a flow time of T throughout the sand. c_0 is the difference of concentrations of the water concentration in the fluid body and the sand reference concentration. Hence after infinite time spend in the sand a fluid parcel has $c_0 = 0$.

The boundaries of the integral mark the regional change from inflow into the sand to outflow, which can be seen from the sign change of $\boldsymbol{u}_D \cdot \boldsymbol{n} \approx \cos(x+\theta)$.

We hence get an expression for the particle increase added to the fluid body from washing out.

$$\frac{\mathrm{d}\, n_0}{\mathrm{d}\, t} = \int_{\pi/2-\theta}^{3\pi/2-\theta} \Delta c\left(T\left(x^*\right), c_0\right) \underbrace{\left(-A\cos\left(x^*\right)\right)}_{\Phi(x^*)} \ \mathrm{d}\, x^* \tag{8}$$

If we now assume an exponential exchange with a constant concentration of c_s i.e. $\Delta c\left(T, c_0\right) = \left(\mathrm{e}^{-\gamma T} - 1\right) c_0$ we can determine the first order macroscopic exchange rate. To do this note that for small ϵ we have small A and hence large $T\left(x^*\right)$. This means that to leading order $\Delta c \approx 1$ and hence we get the total flux.

$$\frac{\mathrm{d}\, n_0}{\mathrm{d}\, t} \approx \int_{\pi/2-\theta}^{3\pi/2-\theta} \left(c_s - c_0\right)\left(-A\cos\left(x^*\right)\right) \ \mathrm{d}\, x^* = -A\cos(\theta)c_0 \tag{9}$$

Note that c_0 and n_0 are closely related as the describe the concentration and the particle number, respectively, in the fluid body.

Now in the case of purely potential flow we have obtained the results as follows:

$$\theta = 0 \tag{10a}$$

$$A = \mathcal{D}a\mathcal{R}e\beta\epsilon = \kappa k^2 \frac{U}{k\nu}\left(1 + \frac{g}{U^2 k}\right)\epsilon = \frac{\kappa k U \epsilon}{\nu} + \frac{\kappa g \epsilon}{\nu U} = \left(kU^2 + g\right)\frac{\kappa\epsilon}{\nu U} \tag{10b}$$

This in particular implies that A is equal to the global exchange rate to leading order.

For this result to be accurate we need: $\mathcal{R}e \gg 1$, $\mathcal{D}a \ll 1$ and $\epsilon \ll 1$. But also $\mathcal{R}e$ cannot be to big as otherwise we get recirculation, turbulence or even sand liquefaction. Unfortunately the pressure is not linear in the sinusoidal disturbance and hence this does not provide a method for arbitrary disturbances.

Hence the final result is in dimensional form:

$$\frac{\mathrm{d}\, n_0}{\mathrm{d}\, t} \approx -\left(kU^2 + g\right)\frac{\kappa\epsilon}{\nu U}\cdot c_0 \tag{11}$$

5 Streamline coordinates

Introduce new coordinates defined by the streamfunction $\psi(x, y)$ and the potential function $\phi(x, y)$. These present a natural coordinates system for developing a boundary layer see Benjamin [1959]. First the equations are developed. This is merely a algebraically difficult task and hence only the results are stated in form of the vorticity equation, (12), and in form of the momentum equation, (13). An expression for the tangential shear stress is also found (15). The boundary condition are given by mass conservation in normal direction across the boundary (14a) and the Beavers-Joseph condition on the tangential shear stress and tangential velocity (14b).

$$\left(\omega_\psi \frac{\partial}{\partial\phi} - \omega_\phi \frac{\partial}{\partial\psi}\right)\left(\mathcal{J}\nabla^2\omega\right) = \mathcal{R}e^{-1}\mathcal{J}\nabla^4\omega, \tag{12a}$$

$$u = \sqrt{\mathcal{J}}\omega_\psi, \qquad v = -\sqrt{\mathcal{J}}\omega_\phi, \tag{12b}$$

$$\mathcal{J} = {\phi_x}^2 + {\phi_y}^2 = \phi_x\psi_y - \phi_y\psi_x = {\psi_x}^2 + {\psi_y}^2 \tag{12c}$$

$$\mathcal{J}\left(\omega_\psi\omega_{\phi\psi} - \omega_\phi\omega_{\psi\psi}\right) + \frac{1}{2}\mathcal{J}_\phi\left({\omega_\phi}^2 + {\omega_\psi}^2\right) = -p_\phi + \mathcal{R}e^{-1}\left(\mathcal{J}\nabla^2\omega\right)_\psi \tag{13a}$$

$$\mathcal{J}\left(\omega_\phi\omega_{\phi\psi} - \omega_\psi\omega_{\phi\phi}\right) + \frac{1}{2}\mathcal{J}_\psi\left({\omega_\phi}^2 + {\omega_\psi}^2\right) = -p_\psi - \mathcal{R}e^{-1}\left(\mathcal{J}\nabla^2\omega\right)_\phi \tag{13b}$$

$$\tag{13c}$$

$$v|_{\psi=0} = v_D|_{\psi=0} = -\left.\mathcal{R}e^{-1}\mathcal{D}a\sqrt{\mathcal{J}}p_\psi\right|_{\psi=0}, \tag{14a}$$

$$\tau|_{\psi=0} = \mathcal{R}e\mathcal{D}a^{-1/2}\left.\left(u - u_D\right)\right|_{\psi=0} = \mathcal{R}e\mathcal{D}a^{-1/2}\left.\left(u + \mathcal{R}e^{-1}\mathcal{D}a\sqrt{\mathcal{J}}p_\phi\right)\right|_{\psi=0} \tag{14b}$$

$$\tau = \mathcal{R}e^{-1}\left(\left({\phi_x}^2 - {\phi_y}^2\right)\left(\omega_{\psi\psi} - \omega_{\phi\phi}\right) + 4\phi_x\phi_y\omega_{\phi\psi} - 2\psi_{xy}\omega_\phi + 2\phi_{xy}\omega_\psi\right) \tag{15a}$$

$$\tau \approx \mathcal{R}e^{-1}\left({\phi_x}^2 - {\phi_y}^2\right)\omega_{\psi\psi} \tag{15b}$$

Note that in these coordinates the irrotational flow solution is simply $\omega = \psi, p = -J/2$.

Now the boundary layer solution is sought as a perturbation, i.e. assume a new streamfunction $\omega = \psi + \tilde{\omega}$. Now two assumptions are made:

- the standard boundary layer assumption that gradients normal to the boundary are much larger than gradients perpendicular to it.

$$\frac{\partial}{\partial \phi} \ll \frac{\partial}{\partial \psi} \tag{16}$$

- the assumption that the perturbation is small

$$\tilde{\omega} \ll \psi \tag{17}$$

The first assumption is well established and standardly used in boundary layer type solution. The second assumption however imposes a bigger problem: as we approach the boundary the perturbation grows and eventually can no longer be assumed small, especially as the irrotational streamfunction goes to 0. Hence this makes it impossible to impose the no slip condition, for example. However, as shall be seen the Beavers-Joseph condition imposes far less perturbation from the irrotational solution and hence can be imposed. Another limitation is that only one boundary condition of the two physically necessary can be imposed in this way.

The full system used for the boundary layer is:

$$\zeta_\phi = \mathcal{R}e^{-1}\zeta_{\psi\psi} \tag{18a}$$
$$\zeta = \mathcal{J}\omega_{\psi\psi} \tag{18b}$$
$$p_\phi = \mathcal{R}e^{-1}\zeta_\psi - \left(\mathcal{J}\omega_\psi\right)_\phi \tag{18c}$$
$$\zeta \to 0 \quad \text{as} \quad \psi \to \infty \tag{18d}$$
$$\zeta(\phi, \psi) = \zeta(\phi + 2\pi, \psi) \tag{18e}$$
$$\frac{{\phi_x}^2 - {\phi_y}^2}{\sqrt{\mathcal{J}}}\, \omega_{\psi\psi}|_{\psi=0} = \frac{\mathcal{R}e^2}{\mathcal{D}a^{1/2}}\, \omega_\psi|_{\psi=0} + \mathcal{R}e\mathcal{D}a^{1/2}\, p_\phi|_{\psi=0} \tag{18f}$$

This is a simple diffusion equation in the vorticity ζ and hence can be solved by separation of variables. Imposing both periodicity (**??**) and decay (**??**) we can get the general series solution:

$$\zeta = \sum_{n=1}^{\infty} \left(A_n \cos\left(n\phi - \sqrt{n\mathcal{R}e/2}\psi\right) + B_n \sin\left(n\phi - \sqrt{n\mathcal{R}e/2}\psi\right)\right) \mathrm{e}^{-\sqrt{n\mathcal{R}e/2}\psi} = \tag{19}$$

Notes for further progress:

- integrate equation to get p as series (IbPs)
- integrate ζ equation once to replace ω_ψ as series in BC (show principle)
- plug in series and equate terms (first order only)
- get solution for p
- integrate ζ twice to get formula for ω

6 Stokes Flow

For Stokes flow we can make use of the similarity of the Darcy-Brinkmann-Forcheimer equation and the Navier-Stokes equation, as in this case in both regions the Reynolds number is small. Using a similar formulation as in we find following system:

$$\varepsilon\frac{\partial \boldsymbol{u}}{\partial t} + \varepsilon^2 \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \tilde{\mu}\nabla^2\boldsymbol{u} - B\frac{\mu}{k}\boldsymbol{u} \tag{20a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{20b}$$

Employing a stream funciton ψ, non-dimensionalising and ignoring inertial terms, i.e. assuming stokes flow, we get a simple equation incorporating a discontinuous function $\alpha(x, y)$:

$$\nabla^4\psi = \alpha\nabla^2\psi \tag{21}$$

Notes for further progress:

- no-slip BC at given depth within sand
- prescribe velocity at some height above sand
- periodicity BCs at vertical boundaries
- solve in rectangular domain
- note: α is non zero constant in porous region and zero in the fluid

A Variables

Basic Flow Variables		
Variable	Dimensionalisation	Description
$\boldsymbol{u}$	U	velocity of the fluid
ϕ	U/k	velocity potential, $\boldsymbol{u} = \boldsymbol{\nabla}\phi$
ψ	U/k	streamfunction, $\boldsymbol{u} = (\psi_y, -\psi_x)$
p	ρU^2	pressure

Physical Parameters	
Variable	Description
ϵ	waveslope
k	wavenumber
U	exterior velocity
ρ	density
μ	kinematic viscosity
ν	dynamic viscosity
g	gravity
κ	permeability of the sediment
γ	exchange coefficient
c_0	concentration difference between the sand and the fluid

Non-dimensional Quantities			
Variable		Formula	Description
$\mathcal{R}e$	$=$	$U/k\nu$	Reynolds number
$\mathcal{D}a$	$=$	κk^2	Darcy number
β	$=$	$\left(1 + \frac{g}{U^2 k}\right)$	
A	$=$	$\mathcal{D}a\mathcal{R}e\beta\epsilon$	

Path Integration Quantities		
Variable	Dimensionalisation	Description
Γ	U/k	infinitesimal path along sediment boundary
γ	U/k	flow path through sediment
$\boldsymbol{n}$	$1/k$	normal to the sediment
ds	$1/k$	infinitesimal arclength element
$\lambda, d\lambda$	$1/k$	alternative parametrisation (in x-direction)

Variables related to Boundary Layer Approach		
Variable	Dimensionalisation	Description
ω	U/k	streamfunction of viscos corrected flow
$\tilde{\omega}$	U/k	perturbation to inviscid streamfunction
$\mathcal{J}$	U^2	jacobian
τ	ρU^2	stress
ζ	Uk	vorticity

Variables related to Stokes Flow Approach	
Variable	Description
ε	porousity
$\tilde{\mu}$	effective viscosity
B	binary parameter

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